L27 March 19 Invariant

Thursday, March 19, 2015 9:47 AM

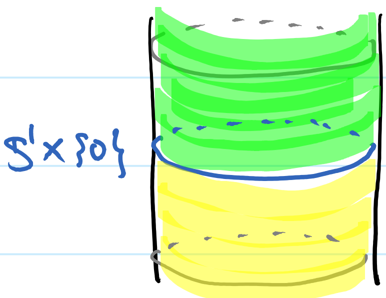
Homeomorphic or not

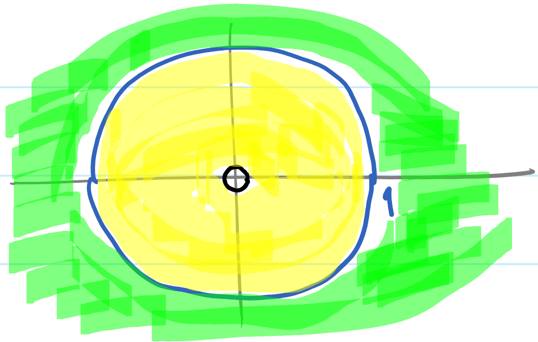
To prove $X = Y$ (meaning homeomorphic)

basically construct $f: X \longrightarrow Y$

$$S^1 \times (-\infty, \infty) \longrightarrow \mathbb{R}^2 \setminus \{0\}$$

$$(e^{i\theta}, t) \longmapsto (e^{t} \cos\theta, e^{t} \sin\theta)$$





But, to prove $X \neq Y$

cannot check all possible mappings

① $(\mathbb{R}, \text{standard}) \neq (S^1, \text{standard})$

why?

Answer. non-compact compact

② $([0,1], \text{standard}) \neq (S^1, \text{standard})$

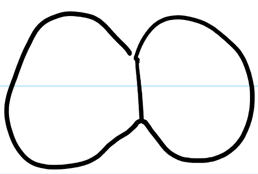
why?

$\exists\, x_0 \in X$, $X \setminus \{x_0\}$ disconnected

$\forall\, y_0 \in Y$ $Y \setminus \{y_0\}$ connected

Exercise

$X \neq Y$

Exercise.  $\neq S^1$

General Principle.

Find a topological property P, i.e., if X satisfies P then its homeomorphic image also satisfies P

① P: compactness

② P: $\exists\, x_0 \in X$ such that $X \setminus \{x_0\}$ is disconnected

In another form

① Define $k(X) = \begin{cases} 1 & \text{if } X \text{ is compact} \\ -1 & \text{if } X \text{ is non-compact} \end{cases}$

Fact. $X \cong Y \implies k(X) = k(Y)$

② Define $c(X) = \#$ of connected component

Clearly, $X \cong Y \implies c(X) = c(Y)$

But $c([0,1]) = 1 = c(S^1)$ no conclusion

Let $s(X) = \sup\{c(X \setminus \{x\}) : x \in X\}$

$$s([0,1]) = 2 \neq s(S^1) = 1$$

Fact. $X \cong Y \implies s(X) = s(Y)$

Topological Invariant

Function $\{\text{Topological spaces}\} \xrightarrow{\ \iota\ } \{\text{Mathematical objects}\}$

numbers, matrices
polynomials
vector spaces, etc.

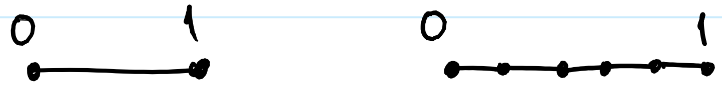
satisfying
$X \cong Y \implies \iota(X) = \iota(Y)$

Euler Characteristic

$$\{\text{Topological spaces}\} \xrightarrow{\chi} \mathbb{Z}$$

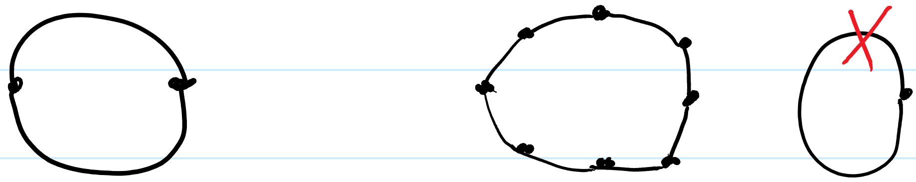
$$\chi(X) \overset{\text{Roughly}}{=\!=} \begin{cases} V-E & \text{if } X \text{ is 1-dim} \\ V-E+F & \text{if } X \text{ is 2-dim} \\ \sum_{k=0}^{n} (-1)^k b_k & \text{if } X \text{ is } n\text{-dim} \end{cases}$$

* $X = [0,1]$



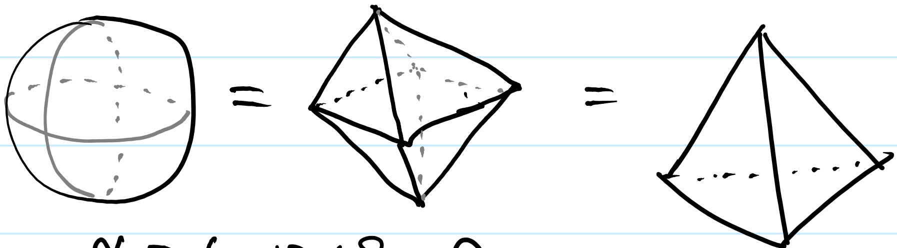
$\chi = 2-1 = 1$ or $6-5=1$

* $X = S^1$

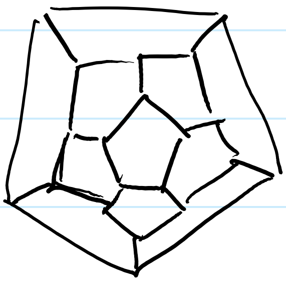


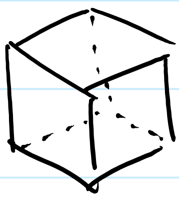
$\chi = 2-2=0$ or $8-8=0$

* $X = S^2$



$\chi = 6-12+8=2$ or $4-6+4=2$





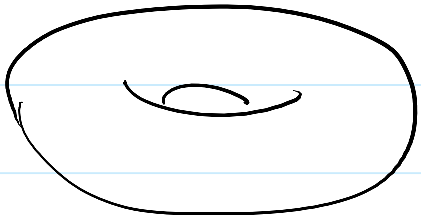
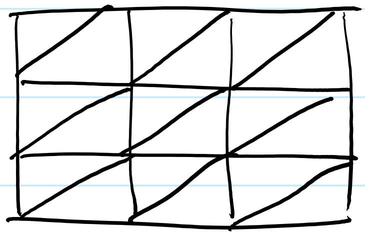
$\chi = 8-12+6=2$

Dodecahedron $= 20-30+12 = 2$

Icosahedron $= 12-30+20 = 2$

* Explore $\chi(\mathbb{R}) = 1$, $\chi(\mathbb{R}^2) = 1$, $\chi(\mathbb{R}^n) = 1$

Torus $\mathbb{T}^2$

 = 

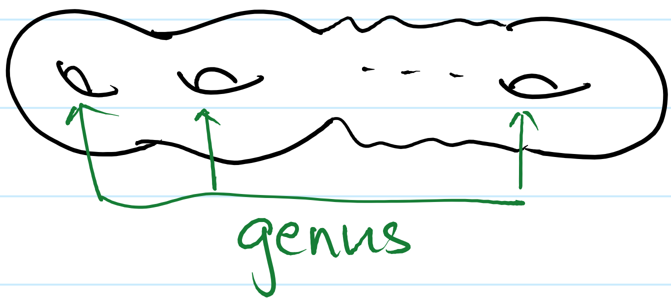
$$\chi(\mathbb{T}^2) = 9 - 27 + 18 = 0$$

$$\mathbb{T}^2 = S^1 \times S^1$$

$$\chi(\mathbb{T}^2) = \chi(S^1) \cdot \chi(S^1)$$

There is algebraic relation on topological invariants

Compact orientable surfaces



$$\chi = 2 - 2g$$

Compact surfaces

$$\chi(\mathbb{P}^2) = 1, \quad \chi(\text{Klein}) = -1$$

Fact. Let X, Y be compact surfaces.

$$X = Y \iff \chi(X) = \chi(Y)$$